

On the estimation of the extreme value index for randomly right-truncated data and application

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Abstract

We introduce a consistent estimator of the extreme value index under random truncation based on a single sample fraction of top observations from truncated and truncation data. We establish the asymptotic normality of the proposed estimator by making use of the weighted tail-copula process framework and we check its finite sample behavior through some simulations. As an application, we provide asymptotic normality results for an estimator of the excess-of-loss reinsurance premium.

Keywords: Bivariate extremes; Hill estimator; Lynden-Bell estimator; Random truncation; Reinsurance premium, Tail dependence.

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1. Introduction

Let $(\mathbf{X}_i, \mathbf{Y}_i)$, $1 \leq i \leq N$, be $N \geq 1$ independent copies from a couple (\mathbf{X}, \mathbf{Y}) of independent positive random variables (rv's) defined over some probability space $(\Omega, \mathcal{A}, \mathbf{P})$, with continuous marginal distribution functions (df's) \mathbf{F} and \mathbf{G} respectively. Suppose that \mathbf{X} is right-truncated by \mathbf{Y} , in the sense that \mathbf{X}_i is only observed when $\mathbf{X}_i \leq \mathbf{Y}_i$. We assume that both survival functions $\overline{\mathbf{F}} := 1 - \mathbf{F}$ and $\overline{\mathbf{G}} := 1 - \mathbf{G}$ are regularly varying at infinity with respective indices $-1/\gamma_1$ and $-1/\gamma_2$. That is, for any $s > 0$

$$\lim_{x \rightarrow \infty} \frac{\overline{\mathbf{F}}(sx)}{\overline{\mathbf{F}}(x)} = s^{-1/\gamma_1} \text{ and } \lim_{y \rightarrow \infty} \frac{\overline{\mathbf{G}}(sy)}{\overline{\mathbf{G}}(y)} = s^{-1/\gamma_2}. \quad (1.1)$$

Being characterized by their heavy tails, these distributions play a prominent role in extreme value theory. They include distributions such as Pareto, Burr, Fréchet, stable and log-gamma, known to be appropriate models for fitting large insurance claims, log-returns, large fluctuations, etc... (see, e.g., [Resnick, 2006](#)). The truncation phenomenon may occur in many fields, for instance, in insurance it is usual that the insurer's claim data do not correspond to the underlying losses, because they are truncated from above. Indeed, when dealing with large claims, the insurance company stipulates an upper limit to the amounts to be paid out. The excesses over this fixed threshold are covered by a reinsurance company. This kind of reinsurance is called excess-loss reinsurance (see, e.g., [Rolski et al., 1999](#)). Depending on the branches of insurance, the upper limit, which may be random, is called in different ways: in life insurance, it is called the cedent's company retention level whereas in non-life insurance, it is called the deductible. For a recent paper on randomly right-truncated insurance claims, one refers to [Escudero and Ortega \(2008\)](#).

Let us now denote (X_i, Y_i) , $i = 1, \dots, n$, to be the observed data, as copies of a couple of rv's (X, Y) with joint df H , corresponding to the truncated sample $(\mathbf{X}_i, \mathbf{Y}_i)$, $i = 1, \dots, N$, where $n = n_N$ is a sequence of discrete rv's. By the law of the large numbers, we have $n_N/N \xrightarrow{\mathbf{P}} p := \mathbf{P}(\mathbf{X} \leq \mathbf{Y})$, as $N \rightarrow \infty$. For convenience, we use, throughout the paper, the notation $n \rightarrow \infty$ to say that $n \xrightarrow{\mathbf{P}} \infty$. For $x, y \geq 0$, we have

$$\begin{aligned} H(x, y) &:= \mathbf{P}(X \leq x, Y \leq y) \\ &= \mathbf{P}(\mathbf{X} \leq x, \mathbf{Y} \leq y \mid \mathbf{X} \leq \mathbf{Y}) = p^{-1} \int_0^x \mathbf{F}(\min(y, z)) d\mathbf{G}(z). \end{aligned}$$

Note that, conditionally on n , the observed data are still independent. The marginal distributions of the observed X 's and Y 's, respectively denoted by F and G , are

equal to

$$F(x) = p^{-1} \int_0^x \overline{\mathbf{G}}(z) d\mathbf{F}(z) \text{ and } G(y) = p^{-1} \int_0^y \mathbf{F}(z) d\mathbf{G}(z),$$

it follows that the corresponding tails

$$\overline{F}(x) = -p^{-1} \int_x^\infty \overline{\mathbf{G}}(z) d\overline{\mathbf{F}}(z) \text{ and } \overline{G}(y) = -p^{-1} \int_y^\infty \mathbf{F}(z) d\overline{\mathbf{G}}(z).$$

It is clear that the asymptotic behavior of \overline{F} simultaneously depends on $\overline{\mathbf{G}}$ and $\overline{\mathbf{F}}$ while that of \overline{G} only relies on $\overline{\mathbf{G}}$. Making use of Potter's bound inequalities (see Lemma 6.3), for the regularly varying functions $\overline{\mathbf{F}}$ and $\overline{\mathbf{G}}$, we may readily show that both \overline{G} and \overline{F} are regularly varying at infinity as well, with respective indices γ_2 and $\gamma := \gamma_1\gamma_2/(\gamma_1 + \gamma_2)$. That is, we have, for any $s > 0$,

$$\lim_{x \rightarrow \infty} \frac{\overline{F}(sx)}{\overline{F}(x)} = s^{-1/\gamma} \text{ and } \lim_{y \rightarrow \infty} \frac{\overline{G}(sy)}{\overline{G}(y)} = s^{-1/\gamma_2}. \quad (1.2)$$

Recently [Gardes and Stupfler \(2014\)](#) addressed the estimation of the extreme value index γ_1 under random truncation. They used the definition of γ to derive the following consistent estimator:

$$\hat{\gamma}_1(k, k') := \frac{\hat{\gamma}(k) \hat{\gamma}_2(k')}{\hat{\gamma}_2(k') - \hat{\gamma}(k)},$$

where

$$\hat{\gamma}(k) := \frac{1}{k} \sum_{i=1}^k \log \frac{X_{n-i+1:n}}{X_{n-k:n}} \text{ and } \hat{\gamma}_2(k') := \frac{1}{k'} \sum_{i=1}^{k'} \log \frac{Y_{n-i+1:n}}{Y_{n-k':n}}, \quad (1.3)$$

are the well-known Hill estimators of γ and γ_2 , with $X_{1:n} \leq \dots \leq X_{n:n}$ and $Y_{1:n} \leq \dots \leq Y_{n:n}$ being the order statistics pertaining to the samples (X_1, \dots, X_n) and (Y_1, \dots, Y_n) respectively. The two sequences $k = k_n$ and $k' = k'_n$ of integer rv's, which satisfy

$$1 < k, k' < n, \quad \min(k, k') \rightarrow \infty \text{ and } \max(k/n, k'/n) \rightarrow 0 \text{ as } n \rightarrow \infty,$$

respectively represent the numbers of top observations from truncated and truncation data. By considering the two situations $k/k' \rightarrow 0$ and $k'/k \rightarrow 0$ as $n \rightarrow \infty$, the authors established the asymptotic normality of $\hat{\gamma}_1(k, k')$, but when $k/k' \rightarrow 1$, they only showed that

$$\sqrt{\min(k, k')} (\hat{\gamma}_1(k, k') - \gamma_1) = O_{\mathbf{P}}(1), \text{ as } n \rightarrow \infty,$$

which is not enough to construct confidence intervals for γ_1 . It is obvious that an accurate computation of the estimate $\hat{\gamma}_1(k, k')$ requires good choices of both k and k' . However from a practical point of view, it is rather unusual in extreme value

analysis to handle two distinct sample fractions simultaneously, which is mentioned by [Gardes and Stupfler \(2014\)](#) in their conclusion as well. For this reason, we consider, in the present work, the situation when $k = k'$ rather than $k/k' \rightarrow 1$. Thus, we obtain an estimator

$$\hat{\gamma}_1 := \hat{\gamma}_1(k) = k^{-1} \frac{\sum_{i=1}^k \log \frac{X_{n-i+1:n}}{X_{n-k:n}} \sum_{i=1}^k \log \frac{Y_{n-i+1:n}}{Y_{n-k:n}}}{\sum_{i=1}^k \log \frac{X_{n-k:n} Y_{n-i+1:n}}{Y_{n-k:n} X_{n-i+1:n}}}, \quad (1.4)$$

of simpler form, expressed in terms of a single sample fraction k of truncated and truncation observations. The number of extreme values used to compute the optimal estimate value $\hat{\gamma}_1$ may be obtained by applying one of the various heuristic methods available in the literature such that, for instance, the algorithm of page 137 in [Reiss and Thomas \(2007\)](#), which will be applied in Section 3.

The task of establishing the asymptotic normality of $\hat{\gamma}_1$ is a bit delicate as one has to take into account the dependence structure of X and Y . The authors of [Gardes and Stupfler \(2014\)](#) avoided this issue by putting conditions on the sample fractions k and k' . In our case we require that the joint df H have a stable tail dependence function ℓ (see [Huang, 1992](#) and [Drees and Huang, 1998](#)), in the sense that the following limit exists:

$$\lim_{t \downarrow 0} t^{-1} \mathbf{P} \left(\overline{F}(X) \leq tx \text{ or } \overline{G}(Y) \leq ty \right) =: \ell(x, y), \quad (1.5)$$

for all $x, y \geq 0$ such that $\max(x, y) > 0$. Note that the corresponding tail copula function is defined by

$$\lim_{t \downarrow 0} t^{-1} \mathbf{P} \left(\overline{F}(X) \leq tx, \overline{G}(Y) \leq ty \right) =: R(x, y), \quad (1.6)$$

which equals $x + y - \ell(x, y)$. In other words, we assume that H belongs to the domain of attraction of a bivariate extreme value distribution. This may be split into two sets of conditions, namely conditions for the convergence of the marginal distributions (1.2) and others for the convergence of the dependence structure (1.5). For details on this topic, see for instance Section 6.1.2 of [de Haan and Ferreira \(2006\)](#) and the papers of [Huang \(1992\)](#), [Schmidt and Stadtmüller \(2006\)](#), [Einmahl et al. \(2006\)](#), [de Haan et al. \(2008\)](#) and [Peng \(2010\)](#).

The rest of the paper is organized as follows. In Section 2, we give our main result which consists in a Gaussian approximation to $\hat{\gamma}_1$ only by assuming the second-order

conditions of regular variation and the stability of the tail dependence function. A simulation study is carried out, in Section 3, to illustrate the performance of $\hat{\gamma}_1$. Section 4 is devoted to an application, as we derive the asymptotic normality of an excess-of-loss reinsurance premium estimator. Finally, the proofs are postponed to Section 5 whereas some results that are instrumental to our needs are gathered in the Appendix.

2. Main results

Weak approximations of extreme value theory based statistics are achieved in the second-order framework (see de Haan and Stadtmüller, 1996). Thus, it seems quite natural to suppose that both df's F and G satisfy the well-known second-order condition of regular variation. That is, we assume that for any $x > 0$

$$\begin{aligned} \lim_{z \rightarrow \infty} \frac{1}{A(z)} \left(\frac{U(zx)}{U(z)} - x^\gamma \right) &= x^\gamma \frac{x^\tau - 1}{\tau}, \\ \lim_{z \rightarrow \infty} \frac{1}{A_2(z)} \left(\frac{U_2(zx)}{U_2(z)} - x^{\gamma_2} \right) &= x^{\gamma_2} \frac{x^{\tau_2} - 1}{\tau_2}, \end{aligned} \quad (2.7)$$

where $U := (1/\bar{F})^\leftarrow$, $U_2 := (1/\bar{G})^\leftarrow$ (with $E^\leftarrow(u) := \inf\{v : E(v) \geq u\}$, for $0 < u < 1$, denoting the quantile function pertaining to a function E), $|A|$ and $|A_2|$ are some regularly varying functions with negative indices (second-order parameters) τ and τ_2 respectively.

Theorem 2.1. *Assume that the second-order regular variation condition (2.7) and (1.5) hold. Let $k := k_n$ be a sequence of integers such that $k \rightarrow \infty$, $k/n \rightarrow 0$, $\sqrt{k}A(n/k) = O(1) = \sqrt{k}A_2(n/k)$. Then, there exist two standard Wiener processes $\{W_i(t), t \geq 0\}$, $i = 1, 2$, defined on the probability space $(\Omega, \mathcal{A}, \mathbf{P})$ with covariance function $R(\cdot, \cdot)$, such that*

$$\sqrt{k} \left(\frac{X_{n-k:n}}{U(n/k)} - 1 \right) - \gamma W_1(1) = o_{\mathbf{P}}(1) = \sqrt{k} \left(\frac{Y_{n-k:n}}{U_2(n/k)} - 1 \right) - \gamma_2 W_2(1),$$

and

$$\begin{aligned} &\sqrt{k}(\hat{\gamma}_1 - \gamma_1) - \mu(k) \\ &= \int_0^1 t^{-1} (cW_1(t) - c_2W_2(t)) dt - cW_1(1) + c_2W_2(1) + o_{\mathbf{P}}(1), \end{aligned}$$

where $c := \gamma_1^2/\gamma$, $c_2 := \gamma_1^2/\gamma_2$ and

$$\mu(k) := \frac{c\sqrt{k}A(n/k)}{\gamma(1-\tau)} + \frac{c_2\sqrt{k}A_2(n/k)}{\gamma_2(1-\tau_2)}.$$

Corollary 2.1. *Under the assumptions of Theorem 2.1, we have*

$$\sqrt{k}(\hat{\gamma}_1 - \gamma_1) \xrightarrow{\mathcal{D}} \mathcal{N}(\mu, \sigma^2), \text{ as } n \rightarrow \infty,$$

provided that $\sqrt{k}A(n/k) \rightarrow \lambda$ and $\sqrt{k}A_2(n/k) \rightarrow \lambda_2$, where

$$\mu := \frac{c\lambda}{\gamma(1-\tau)} + \frac{c_2\lambda_2}{\gamma_2(1-\tau_2)} \text{ and } \sigma^2 := 2c^2 + 2c_2^2 - 2cc_2\delta,$$

with

$$\delta = \delta(R) := \int_0^1 \int_0^1 \frac{R(s,t)}{st} ds dt - \int_0^1 (R(s,1) - R(1,s)) ds + R(1,1).$$

Remark 2.1. *Note that σ^2 is finite. Indeed, the fact that $\ell(x, y)$ is a tail copula function, implies that $\max(x, y) \leq \ell(x, y) \leq x + y$ (see, e.g., [Gudendorf and Segers, 2010](#)) and since $R(x, y) = x + y - \ell(x, y)$, then $0 \leq R(x, y) \leq \min(x, y)$. It follows that*

$$\int_0^1 \int_0^1 \frac{R(s,t)}{st} ds dt \leq \int_0^1 \int_0^1 \frac{\min(s,t)}{st} ds dt = 2,$$

$$\int_0^1 R(s,1) ds \leq \frac{1}{2}, \quad \int_0^1 R(1,s) ds \leq \frac{1}{2} \text{ and } R(1,1) \leq 1.$$

Therefore $|\delta| \leq 4$, which yields that $\sigma^2 < \infty$.

The following corollary directly leads to a practical construction of confidence intervals for the tail index γ_1 .

Corollary 2.2. *Under the assumptions of Corollary 2.1, we have*

$$\frac{\sqrt{k}(\hat{\gamma}_1 - \gamma_1) - \hat{\mu}}{\hat{\sigma}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, 1), \text{ as } n \rightarrow \infty,$$

where $\hat{\mu} = \frac{\hat{c}\hat{\lambda}}{\hat{\gamma}(1-\hat{\tau})} + \frac{\hat{c}_2\hat{\lambda}_2}{\hat{\gamma}_2(1-\hat{\tau}_2)}$ and $\hat{\sigma}^2 := 2\hat{c}^2 + 2\hat{c}_2^2 - 2\hat{c}\hat{c}_2\hat{\delta}$, with

$$\hat{c} := \hat{\gamma}_1^2/\hat{\gamma}, \quad \hat{c}_2 := \hat{\gamma}_1^2/\hat{\gamma}_2, \quad \hat{\delta} := \delta(\hat{R}),$$

$$\hat{\lambda} := \sqrt{k}\hat{\tau} \frac{X_{n-2k:n} - 2^{-\hat{\gamma}}X_{n-k:n}}{2^{-\hat{\gamma}}(2^{-\hat{\tau}} - 1)X_{n-k:n}} \text{ and } \hat{\lambda}_2 := \sqrt{k}\hat{\tau}_2 \frac{Y_{n-2k:n} - 2^{-\hat{\gamma}_2}Y_{n-k:n}}{2^{-\hat{\gamma}_2}(2^{-\hat{\tau}_2} - 1)Y_{n-k:n}}.$$

Here $\hat{\gamma}$ and $\hat{\gamma}_2$ are the respective Hill estimators of γ and γ_2 defined in (1.3) with $k' = k$, $\hat{\tau}$ (resp. $\hat{\tau}_2$) is one of the estimators of τ (resp. τ_2) (see, e.g., [Gomes and Pestana, 2007](#)) and \hat{R} is a nonparametric estimator of R given in [Peng \(2010\)](#) by $\hat{R}(s, t) := k^{-1} \sum_{i=1}^n \mathbf{1}(X_i \geq X_{n-[ks]:n}, Y_i \geq Y_{n-[kt]:n})$, with $[x]$ standing for the integer part of the real number x and $\mathbf{1}(\cdot)$ for the indicator function.

3. SIMULATION STUDY

We carry out a simulation study to illustrate the performance of our estimator, through two sets of truncated and truncation data, both drawn from Burr's model. We have

$$\mathbf{F}(x) = 1 - (1 + x^{1/\delta})^{-\delta/\gamma_1} \text{ and } \mathbf{G}(y) = 1 - (1 + y^{1/\delta})^{-\delta/\gamma_2}, \quad x, y > 0,$$

with $\delta > 0$ and $0 < \gamma_1 < \gamma_2$. The second-order parameters of (2.7) are $\tau = -2\gamma/\delta$ and $\tau_2 = -\gamma_2/\delta$. The truncation probability is equal to $1 - p$ with $p = \gamma_2/(\gamma_1 + \gamma_2)$. We fix $p = 0.7, 0.8, 0.9$ and $\gamma_1 = 0.6, 0.8$. The corresponding γ_2 -values are obtained by solving the latter equation. We vary the common size N of both samples and for each size, we generate 200 independent replicates. Our overall results are then taken as the empirical means of the values obtained in the 200 repetitions. To determine the optimal number of upper order statistics used in the computation of $\hat{\gamma}_1$, we apply the algorithm of page 137 in [Reiss and Thomas \(2007\)](#).

This study consists in two parts: point estimation and 95%—confidence interval construction. In the first part, we evaluate the bias and the root of the mean squared error (rmse) of $\hat{\gamma}_1$ while in the second, we investigate the accuracy of the confidence intervals of the tail index γ_1 , by computing their lengths and coverage probabilities (denoted by ‘covpr’). The results of the first part are summarized in Table 3.1, whereas those of the second are given in Table 3.2, where ‘lcb’ and ‘ucb’ respectively stand for the lower and upper confidence bounds. To compute confidence bounds for γ_1 , with level $(1 - \zeta) \times 100\%$ where $0 < \zeta < 1$, from two realizations (x_1, \dots, x_n) and (y_1, \dots, y_n) of (X_1, \dots, X_n) and (Y_1, \dots, Y_n) respectively, we use Corollary 2.2 and proceed as follows.

- Select the optimal sample fraction of top statistics that we denote by k^* .
- Compute the corresponding $\gamma_1^* = \hat{\gamma}_1(k^*)$, $\gamma_2^* = \hat{\gamma}_2(k^*)$, $\gamma^* = \hat{\gamma}(k^*)$, $c^* = \hat{c}(k^*)$ and $c_2^* = \hat{c}_2(k^*)$.
- Calculate $\tau^* = \hat{\tau}(k^*)$ and $\tau_2^* = \hat{\tau}_2(k^*)$ via one of the available numerical procedures (see, e.g., [Gomes and Pestana, 2007](#)) and then get $\lambda^* = \hat{\lambda}(k^*)$, $\lambda_2^* = \hat{\lambda}_2(k^*)$.
- Evaluate $\delta^* = \hat{\delta}(k^*)$ by means of Monte Carlo integration.
- Compute $\mu^* = \hat{\mu}(k^*)$ and $\sigma^* = \hat{\sigma}(k^*)$.

$p = 0.70$										
N	n	k	$\hat{\gamma}_1$	bias	rmse	n	k	$\hat{\gamma}_1$	bias	rmse
500	350	15	0.515	-0.084	0.299	349	15	0.673	-0.127	0.356
1000	701	35	0.555	-0.044	0.264	699	32	0.704	-0.095	0.307
1500	1049	50	0.554	-0.046	0.212	1049	51	0.751	-0.049	0.259
$p = 0.80$										
500	400	18	0.521	-0.079	0.233	400	18	0.723	-0.077	0.351
1000	801	43	0.566	-0.034	0.181	799	40	0.713	-0.087	0.273
1500	1198	64	0.566	-0.033	0.145	1200	64	0.752	-0.048	0.203
$p = 0.90$										
500	450	22	0.547	-0.053	0.186	449	20	0.702	-0.098	0.295
1000	900	45	0.558	-0.042	0.148	900	49	0.747	-0.053	0.189
1500	1349	76	0.577	-0.023	0.118	1348	77	0.755	-0.045	0.151

TABLE 3.1. Point estimation of the tail index based on 200 samples from randomly right-truncated Burr population with shape parameter 0.6 (left panel) and 0.8 (right panel)

At last, the $(1 - \zeta) \times 100\%$ -confidence bounds for the extreme value index γ_1 are

$$\gamma_1^* + \frac{1}{\sqrt{k^*}} (\mu^* \pm \sigma^* z_{\zeta/2}),$$

where $z_{\zeta/2}$ is the $(1 - \zeta/2)$ -quantile of the standard normal rv.

On the light of the results of both tables, we see that truncation is the factor that affects most the estimation process of the tail index. As we would have expected, the smaller the truncation percentage is, the better and more accurate the estimation is, for both index values and each sample size. The reason why we don't consider small samples (we start with a size of 500) is that, in extreme-value theory based inference, large samples are needed in order for the results to be significant. This motivation becomes more obvious when, in addition, there is truncation.

4. Application: excess-of-loss reinsurance premium estimation

As an application of Theorem 2.1, we derive the asymptotic normality of an estimator of the excess-of-loss reinsurance premium obtained with truncated data. Our choice is motivated mainly by two reasons. The first one is that reinsurance is a very important field of application of extreme value theory and the second is that

$p = 0.70$						
N	lcb–ucb	covpr	length	lcb–ucb	covpr	length
500	0.226 – 0.993	0.94	0.767	0.294 – 1.199	0.92	0.905
1000	0.319 – 0.851	0.94	0.532	0.428 – 1.099	0.91	0.671
1500	0.396 – 0.801	0.90	0.405	0.516 – 1.043	0.89	0.527
$p = 0.80$						
500	0.242 – 0.880	0.93	0.638	0.313 – 1.158	0.93	0.845
1000	0.376 – 0.790	0.92	0.414	0.497 – 1.041	0.92	0.544
1500	0.436 – 0.759	0.91	0.323	0.576 – 1.003	0.91	0.427
$p = 0.90$						
500	0.317 – 0.820	0.90	0.503	0.438 – 1.085	0.92	0.647
1000	0.408 – 0.750	0.91	0.342	0.540 – 0.998	0.90	0.458
1500	0.445 – 0.727	0.90	0.282	0.591 – 0.965	0.90	0.374

TABLE 3.2. Accuracy of 95%-confidence intervals for the tail index based on 200 samples from randomly right-truncated Burr population with shape parameter 0.6 (left panel) and 0.8 (right panel)

data sets with truncated extreme observations may very likely be encountered in insurance. The aim of reinsurance, where emphasis lies on modelling extreme events, is to protect an insurance company, called ceding company, against losses caused by excessively large claims and/or a surprisingly high number of moderate claims. Nice discussions on the use of extreme value theory in the actuarial world (especially in the reinsurance industry) can be found, for instance, in [Embrechts *et al.* \(1997\)](#), a major textbook on the subject, and [Beirlant *et al.* \(2004\)](#).

Let $\mathbf{X}_1, \dots, \mathbf{X}_n$ ($n \geq 1$) be n individual claim amounts of an insured heavy-tailed loss \mathbf{X} with finite mean. A Pareto-like distribution, with tail index greater than or equal to 1, does not have finite mean. Hence, assuming that $\mathbf{E}[\mathbf{X}]$ exists necessarily implies that $\gamma_1 < 1$. In the excess-of-loss reinsurance treaty, the ceding company covers claims that do not exceed a (high) number $u \geq 0$, called retention level, while the reinsurer pays the part $(\mathbf{X}_i - u)_+ := \max(0, \mathbf{X}_i - u)$ of each claim exceeding u . The net premium for the layer from u to infinity is defined as follows:

$$\Pi = \Pi(u) := \mathbf{E}[(\mathbf{X} - u)_+] = \int_u^\infty \bar{\mathbf{F}}(x) dx,$$

which may be rewritten into $\Pi = u\overline{\mathbf{F}}(u) \int_1^\infty \overline{\mathbf{F}}(ux) / \overline{\mathbf{F}}(u) dx$. By using the well-known Karamata theorem (see, for instance, Theorem B.1.5 in [de Haan and Ferreira, 2006](#), page 363) we have, for large u ,

$$\Pi \sim \frac{\gamma_1}{1 - \gamma_1} u \overline{\mathbf{F}}(u), \quad 0 < \gamma_1 < 1.$$

As we see, a semi-parametric estimator for \mathbf{F} is needed in order to estimate the premium Π . To this end, let us define

$$C(x) := \mathbf{P}(\mathbf{X} \leq x \leq \mathbf{Y} \mid \mathbf{X} \leq \mathbf{Y}) = \mathbf{P}(X \leq x \leq Y),$$

with \mathbf{Y} being the truncation rv introduced in Section 1. This quantity C is very crucial as it plays a prominent role in the statistical inference under random truncation. In other words, we have

$$C(x) = p^{-1} \mathbf{F}(x) \overline{\mathbf{G}}(x) = F(x) - G(x) = \overline{G}(x) - \overline{F}(x).$$

It is worth mentioning that, since \mathbf{F} and \mathbf{G} are heavy-tailed then their right endpoints are infinite and thus they are equal. Therefore, from [Woodroffe \(1985\)](#), the functions \mathbf{F} , F and C are linked by

$$C(x) d\mathbf{F}(x) = \mathbf{F}(x) dF(x),$$

known as self-consistency equation (see, e.g., [Strzalkowska-Kominiak and Stute, 2009](#)), whose solution is

$$\mathbf{F}(x) = \exp -\Lambda(x), \quad (4.8)$$

where $\Lambda(x) := \int_x^\infty dF(z) / C(z)$. Replacing F and C by their respective empirical counterparts $F_n(x) := n^{-1} \sum_{i=1}^n \mathbf{1}(X_i \leq x)$ (the usual empirical df based on the fully observed sample (X_1, \dots, X_n)) and $C_n(x) := n^{-1} \sum_{i=1}^n \mathbf{1}(X_i \leq x \leq Y_i)$, yields the well-known Lynden-Bell product limit estimator ([Lynden-Bell, 1971](#)) of \mathbf{F} ,

$$\mathbf{F}_n(x) = \exp -\Lambda_n(x), \quad (4.9)$$

where $\Lambda_n(x) := \int_x^\infty dF_n(z) / C_n(z)$. If there are no ties, \mathbf{F}_n may be put in the form

$$\mathbf{F}_n(x) := \prod_{X_{i:n} > x} \left(1 - \frac{1}{nC_n(X_{i:n})} \right). \quad (4.10)$$

Since $\overline{\mathbf{F}}$ is regularly varying at infinity with index $-1/\gamma_1$, then

$$\overline{\mathbf{F}}(x) \sim \overline{\mathbf{F}}(U(n/k)) (x/U(n/k))^{-1/\gamma_1}, \quad \text{as } x \rightarrow \infty.$$

This leads us to derive a Weissman-type estimator ([Weissman, 1978](#))

$$\widehat{\overline{\mathbf{F}}}(x) = \left(\frac{x}{X_{n-k:n}} \right)^{-1/\widehat{\gamma}_1} \overline{\mathbf{F}}_n(X_{n-k:n}),$$

for the distribution tail $\overline{\mathbf{F}}$ with truncated data. Note that

$$\mathbf{F}_n(X_{n-k:n}) = \prod_{i=n-k+1}^n \left(1 - \frac{1}{nC_n(X_{i:n})} \right).$$

Thus, the distribution tail estimator is of the form

$$\widehat{\overline{\mathbf{F}}}(x) := \left(\frac{x}{X_{n-k:n}} \right)^{-1/\widehat{\gamma}_1} \left\{ 1 - \prod_{i=1}^k \left(1 - \frac{1}{nC_n(X_{n-i+1:n})} \right) \right\}.$$

Consequently, we define an estimator $\widehat{\Pi}_n$ to the premium Π as follows:

$$\widehat{\Pi}_n := \frac{\widehat{\gamma}_1}{1 - \widehat{\gamma}_1} X_{n-k:n} \left(\frac{u}{X_{n-k:n}} \right)^{1-1/\widehat{\gamma}_1} \left\{ 1 - \prod_{i=1}^k \left(1 - \frac{1}{nC_n(X_{n-i+1:n})} \right) \right\}.$$

This estimator coincides with that proposed and applied to the Norwegian fire data by [Beirlant et al. \(2001\)](#), in the non truncation case. Prior to establish the asymptotic normality of $\widehat{\Pi}_n$ ([Theorem 4.2](#)), we give, in the following basic result, an asymptotic representation to the Lynden-bell estimator \mathbf{F}_n (in $X_{n-k:n}$). This result will of prime importance in the study of the limiting behaviors of many statistics based on truncated data exhibiting extreme values.

Theorem 4.1. *Assume that the second-order conditions of regular variation ([2.7](#)) hold with $\gamma_1 < \gamma_2$. Let $k := k_n$ be a sequence of integers such that $k \rightarrow \infty$, $k/n \rightarrow 0$. Then*

$$\sqrt{k} \left(\frac{\overline{\mathbf{F}}_n(X_{n-k:n})}{\overline{\mathbf{F}}(X_{n-k:n})} - 1 \right) = \frac{\gamma_1 \gamma_2}{(\gamma_1 + \gamma_2)^2} \int_0^1 s^{-\gamma/\gamma_2-1} W_1(s) ds + \frac{\gamma_2}{\gamma_1 + \gamma_2} W_1(1) + o_{\mathbf{P}}(1).$$

Consequently,

$$\sqrt{k} \left(\frac{\overline{\mathbf{F}}_n(X_{n-k:n})}{\overline{\mathbf{F}}(X_{n-k:n})} - 1 \right) \xrightarrow{\mathcal{D}} \mathcal{N} \left(0, \frac{\gamma_2^2}{\gamma_2^2 - \gamma_1^2} \right), \text{ as } n \rightarrow \infty.$$

Remark 4.1. *Under the assumptions of [Theorem 4.1](#), we have*

$$\frac{\overline{\mathbf{F}}_n(X_{n-k:n})}{\overline{\mathbf{F}}(X_{n-k:n})} \xrightarrow{\mathbf{P}} 1, \text{ as } n \rightarrow \infty.$$

To establish the asymptotic normality $\widehat{\Pi}_n$, we require the second-order regular variation to \mathbf{F} . That is, we suppose that

$$\lim_{t \rightarrow \infty} \frac{1}{\mathbf{A}(t)} \left(\frac{\overline{\mathbf{F}}(tx)}{\overline{\mathbf{F}}(t)} - x^{-1/\gamma_1} \right) = x^{-1/\gamma_1} \frac{x^{\tau_1/\gamma_1} - 1}{\tau_1 \gamma_1}, \quad (4.11)$$

for any $x > 0$, where $|\mathbf{A}|$ is some regularly varying function at infinity with index τ_1/γ_1 , where $\tau_1 < 0$ is the second-order parameter. For asymptotic theory requirements, one has to specify the relation between the retention level u and the quantile $U(n/k)$. Indeed, as mentioned in [Vandewalle and Beirlant \(2006\)](#), amongst others, extreme value methodology typically applies to u values for which $\mathbf{P}(\mathbf{X} > u) = O(1/n)$, hence $\mathbf{P}(X > u) = O(1/n)$. This leads to situate $u = u_n$ with respect to $U(n/k)$ so that, for large n , the quotient $u/U(n/k)$ tends to some constant a .

Theorem 4.2. *Assume that the second-order regular variation conditions (4.11) hold with $0 < \gamma_1 < 1$ and $\gamma_1 < \gamma_2$. Let $k := k_n$ be a sequence of integers such that $k \rightarrow \infty$, $k/n \rightarrow 0$ and $\sqrt{k}\mathbf{A}(U(n/k)) \rightarrow \lambda^* < \infty$. Then, whenever $u/U(n/k) \rightarrow a$, we have as $n \rightarrow \infty$,*

$$\frac{\sqrt{k}(\hat{\Pi}_n - \Pi)}{(u/U(n/k))^{1-1/\gamma_1} U(n/k) \bar{\mathbf{F}}(U(n/k))} \xrightarrow{\mathcal{D}} \mathcal{N}\left(\frac{\lambda^*}{(\gamma_1 - 1 - \tau_1)(\gamma_1 - 1)}, \sigma^{*2}\right),$$

where

$$\sigma^{*2} := \zeta^2 \sigma^2 + \frac{\gamma_1^2 \gamma_2^2}{(\gamma_2^2 - \gamma_1^2)(1 - \gamma_1)^2} + 2 \frac{\gamma \gamma_1 \zeta \delta^*}{1 - \gamma_1},$$

with σ^2 as defined in [Corollary 2.1](#), $\zeta := ((1 - \gamma_1) \log a + \gamma_1) / (\gamma_1 (1 - \gamma_1)^2)$ and

$$\begin{aligned} \delta^* &:= \frac{c}{\gamma \gamma_2} + \frac{c_2}{\gamma_1} \left(R(1, 1) - \int_0^1 \frac{R(1, t)}{t} dt \right) \\ &+ \frac{c_2}{\gamma_1 + \gamma_2} \left(\int_0^1 \frac{R(s, 1)}{s^{\gamma/\gamma_2 + 1}} ds - \int_0^1 \int_0^1 \frac{R(s, t)}{ts^{\gamma/\gamma_2 + 1}} ds dt \right). \end{aligned}$$

5. Proofs

5.1. Proof of Theorem 2.1. We begin by a brief introduction on the weak approximation of a weighed tail copula process given in Proposition 1 of [Einmahl et al. \(2006\)](#). Set $U_i := \bar{F}(X_i)$ and $V_i := \bar{G}(Y_i)$, for $i = 1, \dots, n$, and let $C(x, y)$ be the joint df of (U_i, V_i) . The copula function C and its corresponding tail R , defined in (1.6), are linked by $C(tx, ty) - R(x, y) = O(t^\epsilon)$, as $t \downarrow 0$, for some $\epsilon > 0$, uniformly for $x, y \geq 0$ and $\max(x, y) \leq 1$ ([Huang, 1992](#)). Let us define

$$v_n(x, y) := \sqrt{k}(\mathbf{T}_n(x, y) - R_n(x, y)), \quad x, y > 0,$$

where

$$\mathbf{T}_n(x, y) := \frac{1}{k} \sum_{i=1}^n \mathbf{1} \left(U_i < \frac{k}{n}x, V_i < \frac{k}{n}y \right) \quad \text{and} \quad R_n(x, y) := \frac{n}{k} C \left(\frac{kx}{n}, \frac{ky}{n} \right).$$

In the sequel, we will need the following two empirical processes:

$$\alpha_n(x) := v_n(x, \infty) = \sqrt{k}(\mathbf{U}_n(x) - x) \text{ and } \beta_n(y) := v_n(\infty, y) = \sqrt{k}(\mathbf{V}_n(y) - y),$$

where

$$\mathbf{U}_n(x) := \mathbf{T}_n(x, \infty) = \frac{1}{k} \sum_{i=1}^n \mathbf{1}\left(U_i < \frac{k}{n}x\right),$$

and

$$\mathbf{V}_n(y) := \mathbf{T}_n(\infty, y) = \frac{1}{k} \sum_{i=1}^n \mathbf{1}\left(V_i < \frac{k}{n}y\right).$$

From assertions (3.8) and (3.9) in [Einmahl et al. \(2006\)](#), there exists a Gaussian process $W_R(x, y)$, defined on the probability space $(\Omega, \mathcal{A}, \mathbf{P})$, with mean zero and covariance

$$\mathbf{E}[W_R(x_1, y_1) W_R(x_2, y_2)] = R(\min(x_1, x_2), \min(y_1, y_2)), \quad (5.12)$$

such that for any $M > 0$

$$\sup_{0 < x, y \leq M} \frac{|v_n(x, y) - W_R(x, y)|}{\{\max(x, y)\}^\eta} = o_{\mathbf{P}}(1),$$

and

$$\sup_{0 < x \leq M} \frac{|\alpha_n(x) - W_1(x)|}{x^\eta} = o_{\mathbf{P}}(1) = \sup_{0 < y \leq M} \frac{|\beta_n(y) - W_2(y)|}{y^\eta}, \quad (5.13)$$

as $n \rightarrow \infty$, for any $0 \leq \eta < 1/2$, where

$$W_1(x) := W_R(x, \infty) \text{ and } W_2(y) := W_R(\infty, y),$$

are two standard Wiener processes such that $\mathbf{E}[W_1(x) W_2(y)] = R(x, y)$.

To prove our result, we will write the tail index estimator $\hat{\gamma}_1$ in terms of the processes $\alpha_n(\cdot)$ and $\beta_n(\cdot)$. We start by splitting $\hat{\gamma}_1 - \gamma_1$ into the sum of two terms

$$T_{n1} := \frac{\hat{\gamma}_2(\gamma_2 - \gamma) + \gamma_2\gamma}{(\hat{\gamma}_2 - \hat{\gamma})(\gamma_2 - \gamma)}(\hat{\gamma} - \gamma) \text{ and } T_{n2} := -\frac{\gamma^2}{(\hat{\gamma}_2 - \hat{\gamma})(\gamma_2 - \gamma)}(\hat{\gamma}_2 - \gamma_2).$$

Note that, for two sequences of rv's $V_n^{(1)}$ and $V_n^{(2)}$, we use the notation $V_n^{(1)} \approx V_n^{(2)}$ to say that $V_n^{(1)} = V_n^{(2)}(1 + o_{\mathbf{P}}(1))$, as $n \rightarrow \infty$. Since both $\hat{\gamma}$ and $\hat{\gamma}_2$ are consistent estimators ([Mason, 1982](#)), then, as $n \rightarrow \infty$, we have

$$T_{n1} \approx \frac{c}{\gamma}(\hat{\gamma} - \gamma) \text{ and } T_{n2} \approx -\frac{c_2}{\gamma_2}(\hat{\gamma}_2 - \gamma_2),$$

where c_1 and c_2 are those defined in Theorem 2.1. In other words, we have, as $n \rightarrow \infty$,

$$\sqrt{k}(\hat{\gamma}_1 - \gamma_1) \approx \frac{c}{\gamma}\sqrt{k}(\hat{\gamma} - \gamma) - \frac{c_2}{\gamma_2}\sqrt{k}(\hat{\gamma}_2 - \gamma_2) \quad (5.14)$$

Next, we represent $\sqrt{k}(\hat{\gamma} - \gamma)$ and $\sqrt{k}(\hat{\gamma}_2 - \gamma_2)$ in terms of $\alpha_n(\cdot)$ and $\beta_n(\cdot)$ respectively. For the first term, we use the first-order condition of regular variation of \bar{F} (1.2) and apply Theorem 1.2.2 in [de Haan and Ferreira \(2006\)](#) to have

$$\lim_{n \rightarrow \infty} \frac{n}{k} \int_{F^{\leftarrow}(1-k/n)}^{\infty} t^{-1} \bar{F}(t) dt = \gamma,$$

this allows us to write $\hat{\gamma} = \frac{n}{k} \int_{X_{n-k:n}}^{\infty} t^{-1} \bar{F}_n(t) dt$. Now, we consider the following decomposition $\hat{\gamma} - \gamma = S_{n1} + S_{n2} + S_{n3}$, where

$$S_{n1} := \frac{n}{k} \int_{X_{n-k:n}}^{\infty} t^{-1} (\bar{F}_n(t) - \bar{F}(t)) dt, \quad S_{n2} := -\frac{n}{k} \int_{F^{-1}(1-k/n)}^{X_{n-k:n}} t^{-1} \bar{F}(t) dt$$

and

$$S_{n3} := \frac{n}{k} \int_{F^{-1}(1-k/n)}^{\infty} t^{-1} \bar{F}(t) dt - \gamma.$$

It is easy to verify that, almost surely, we have

$$\bar{F}_n(t) = \frac{k}{n} \mathbf{U}_n\left(\frac{n}{k} \bar{F}(t)\right). \quad (5.15)$$

Without loss of generality and after two successive changes of variables ($u = tX_{n-k:n}$ then $s = n\bar{F}(tX_{n-k:n})/k$), we have

$$S_{n1} = \int_{\frac{n}{k}\bar{F}(X_{n-k:n})}^0 \frac{\mathbf{U}_n(s) - s}{F^{\leftarrow}(1 - sk/n)} dF^{\leftarrow}(1 - sk/n),$$

which we decompose into

$$\begin{aligned} S_{n1} &= \int_1^0 \frac{\mathbf{U}_n(s) - s}{F^{\leftarrow}(1 - sk/n)} dF^{\leftarrow}(1 - sk/n) \\ &\quad + \int_{\frac{n}{k}\bar{F}(X_{n-k:n})}^1 \frac{\mathbf{U}_n(s) - s}{F^{\leftarrow}(1 - sk/n)} dF^{\leftarrow}(1 - sk/n). \end{aligned}$$

For the purpose of using Potter's result of Lemma 6.3 for the quantile function $s \rightarrow F^{\leftarrow}(1 - s)$, we write

$$\begin{aligned} S_{n1} &= \int_1^0 \frac{F^{\leftarrow}(1 - k/n)}{F^{\leftarrow}(1 - sk/n)} (\mathbf{U}_n(s) - s) d\frac{F^{\leftarrow}(1 - sk/n)}{F^{\leftarrow}(1 - k/n)} \\ &\quad + \int_{\frac{n}{k}\bar{F}(X_{n-k:n})}^1 \frac{F^{\leftarrow}(1 - k/n)}{F^{\leftarrow}(1 - sk/n)} (\mathbf{U}_n(s) - s) d\frac{F^{\leftarrow}(1 - sk/n)}{F^{\leftarrow}(1 - k/n)}. \end{aligned}$$

This allows us to write

$$S_{n1} \approx \gamma \left\{ \int_0^1 s^{-1} (\mathbf{U}_n(s) - s) ds - \int_{\frac{n}{k}\bar{F}(X_{n-k:n})}^1 s^{-1} (\mathbf{U}_n(s) - s) ds \right\}.$$

In other words, we have, as $n \rightarrow \infty$,

$$\sqrt{k}S_{n1} \approx \gamma \left\{ \int_0^1 s^{-1} \alpha_n(s) ds - \int_{\frac{n}{k}\bar{F}(X_{n-k:n})}^1 s^{-1} \alpha_n(s) ds \right\}. \quad (5.16)$$

As for the second term S_{n2} , we use the mean value theorem to get

$$S_{n2} = -\frac{n}{k} (X_{n-k:n} - U(n/k)) z_n^{-1} \overline{F}(z_n),$$

where z_n is a sequence of rv's lying between $X_{n-k:n}$ and $U(n/k)$. Observe that we have

$$S_{n2} = -\frac{\overline{F}(z_n)}{\overline{F}(F^{\leftarrow}(1 - k/n))} \frac{U(n/k)}{z_n} \left(\frac{X_{n-k:n}}{U(n/k)} - 1 \right).$$

Since $X_{n-k:n}/U(n/k) \xrightarrow{P} 1$, then $z_n/U(n/k) \xrightarrow{P} 1$ and $\frac{n}{k} \overline{F}(z_n) \xrightarrow{P} 1$. It follows that

$$S_{n2} \approx -\left(\frac{X_{n-k:n}}{U(n/k)} - 1 \right).$$

Recall that $U_i = \overline{F}(X_i)$ and note that $U_{i:n} = \overline{F}(X_{n-i+1:n})$, therefore

$$S_{n2} \approx -\left(\frac{F^{\leftarrow}(1 - U_{k+1:n})}{U(n/k)} - 1 \right).$$

We use Potter's bound inequalities (see Lemma 6.3) together with the mean value theorem to write $S_{n2} \approx \gamma \left(\frac{n}{k} U_{k+1:n} - 1 \right)$. Since $\mathbf{U}_n \left(\frac{n}{k} U_{k+1:n} \right) = 1$, then

$$\sqrt{k} S_{n2} \approx -\gamma \alpha_n \left(\frac{n}{k} U_{k+1:n} \right). \quad (5.17)$$

By summing up (5.16) and (5.17) and making use of the weak approximation (5.13) for $\alpha_n(\cdot)$, we get

$$\begin{aligned} & \sqrt{k} (S_{n1} + S_{n2}) \\ & \approx \gamma \left\{ \int_0^1 s^{-1} W_1(s) ds - \int_{\frac{n}{k} \overline{F}(X_{n-k:n})}^1 s^{-1} W_1(s) ds - W_1\left(\frac{n}{k} U_{k+1:n}\right) \right\}. \end{aligned}$$

Next, we show that $I_n := \int_{\frac{n}{k} \overline{F}(X_{n-k:n})}^1 s^{-1} W_1(s) ds \xrightarrow{P} 0$. For arbitrary $\epsilon, \vartheta > 0$, we write

$$\mathbf{P}(|I_n| > \vartheta) \leq \mathbf{P}\left(\int_{1-\epsilon}^1 s^{-1} W_1(s) ds > \vartheta\right) + \mathbf{P}\left(\left|\frac{n}{k} \overline{F}(X_{n-k:n}) - 1\right| > \epsilon\right).$$

The fact that $\mathbf{E}|W_1(s)| \leq s^{1/2}$ implies that $\mathbf{E}\left|\int_{1-\epsilon}^1 s^{-1} W_1(s) ds\right| \leq \sqrt{2} \left(1 - (1-\epsilon)^{1/2}\right)$. Therefore, by Chebyshev's inequality, we infer that

$$\mathbf{P}\left(\int_{1-\epsilon}^1 s^{-1} W_1(s) ds > \vartheta\right) \leq \sqrt{2} \vartheta^{-2} \left(1 - (1-\epsilon)^{1/2}\right).$$

On the other hand, we have $\frac{n}{k} \overline{F}(X_{n-k:n}) \xrightarrow{P} 1$ this means that for all large n ,

$$\mathbf{P}\left(\left|\frac{n}{k} \overline{F}(X_{n-k:n}) - 1\right| > \epsilon\right) \leq \sqrt{2} \epsilon^{-2} \left(1 - (1-\epsilon)^{1/2}\right).$$

It follows that $\mathbf{P}(|I_n| > \vartheta) \leq \sqrt{2}(\vartheta^{-2} + \epsilon^{-2}) \left(1 - (1 - \epsilon)^{1/2}\right)$ which tends to zero when $\epsilon, \vartheta \downarrow 0$, as sought. Since $nU_{k+1:n}/k \xrightarrow{\mathbf{P}} 1$, then by using similar arguments as the above we have $W_1(nU_{k+1:n}/k) = W_1(1) + o_{\mathbf{P}}(1)$. Consequently, we have

$$\sqrt{k}(S_{n1} + S_{n2}) = (1 + o_{\mathbf{P}}(1)) \gamma \int_0^1 s^{-1} W_1(s) ds - \gamma W_1(1).$$

For the third term, it suffices to use the second-order condition of regular variation to obtain

$$\sqrt{k}S_{n3} = \frac{\sqrt{k}A(n/k)}{1 - \tau} (1 + o(1)) \text{ as } n \rightarrow \infty.$$

In summary, we have

$$\sqrt{k}(\hat{\gamma} - \gamma) = \gamma \int_0^1 s^{-1} W_1(s) ds - \gamma W_1(1) + \frac{\sqrt{k}A(n/k)}{1 - \tau} + o_{\mathbf{P}}(1). \quad (5.18)$$

Likewise, we write $\hat{\gamma}_2 = \frac{n}{k} \int_{Y_{n-k:n}}^{\infty} t^{-1} \overline{G}_n(t) dt$, where $G_n(x) := n^{-1} \sum_{i=1}^n \mathbf{1}(Y_i \leq x)$ is the usual empirical df based on the fully observed sample (Y_1, \dots, Y_n) . Then, by using similar arguments, we express $\hat{\gamma}_2$ in terms of the process $\beta_n(\cdot)$ as follows:

$$\begin{aligned} & \sqrt{k}(\hat{\gamma}_2 - \gamma_2) \\ & \approx \gamma_2 \left\{ \int_0^1 s^{-1} \beta_n(s) ds - \int_{\frac{n}{k} \overline{G}(Y_{n-k:n})}^1 s^{-1} \beta_n(s) ds - \beta_n\left(\frac{n}{k} V_{k+1:n}\right) \right\}. \end{aligned}$$

Then by using approximation (5.13) for $\beta_n(\cdot)$, we obtain

$$\sqrt{k}(\hat{\gamma}_2 - \gamma_2) = \gamma_2 \int_0^1 s^{-1} W_2(s) ds - \gamma_2 W_2(1) + \frac{\sqrt{k}A_2(n/k)}{1 - \tau_2} + o_{\mathbf{P}}(1). \quad (5.19)$$

Finally, substituting results (5.18) and (5.19) in equation (5.14) achieves the proof. \square

5.2. Proof of Corollary 2.1. Elementary calculations, using the covariance formula (5.12) and the fact that $\mathbf{E} \left[\int_0^1 s^{-1} W_i(s) ds \right]^2 = 2$, $i = 1, 2$, straightforwardly lead to the result. \square

5.3. Proof of Corollary 2.2. It suffices to plug the estimate of each parameter in the result of Corollary 2.1. To estimate the limits λ and λ_2 , we exploit the second-order conditions of regular variation (2.7). We have, as $z \rightarrow \infty$,

$$A(z) \sim \tau \frac{U(zx)/U(z) - x^\gamma}{x^\gamma(x^\tau - 1)}, \text{ for any } x > 0.$$

In particular, for $x = 1/2$, and $z = n/k$, we have

$$A(n/k) \sim \tau \frac{U\left(\frac{n}{2k}\right)/U\left(\frac{n}{k}\right) - 2^{-\gamma}}{2^{-\gamma}(2^{-\tau} - 1)}.$$

Hence, we take

$$\hat{A}(n/k) = \hat{\tau} \frac{X_{n-2k:n}/X_{n-k:n} - 2^{-\hat{\gamma}}}{2^{-\hat{\gamma}}(2^{-\hat{\tau}} - 1)} = \hat{\tau} \frac{X_{n-2k:n} - 2^{-\hat{\gamma}}X_{n-k:n}}{2^{-\hat{\gamma}}(2^{-\hat{\tau}} - 1)X_{n-k:n}},$$

an estimate of $A(n/k)$. Thus, the expression of $\hat{\lambda}$ readily follows. The same idea applies to λ_2 as well. \square

5.4. Proof of Theorem 4.1. For convenience we set

$$\mathbf{D}_n^* := \frac{\bar{\mathbf{F}}_n(X_{n-k:n}) - \bar{\mathbf{F}}(X_{n-k:n})}{\bar{\mathbf{F}}(X_{n-k:n})}.$$

Since $\bar{\mathbf{F}}$ is regularly varying at infinity with index $-1/\gamma_1$ and $X_{n-k:n}/U(n/k) \xrightarrow{\mathbf{P}} 1$, then $\bar{\mathbf{F}}(X_{n-k:n}) \approx \bar{\mathbf{F}}(U(n/k))$ and therefore

$$\mathbf{D}_n^* \approx \frac{\bar{\mathbf{F}}_n(X_{n-k:n}) - \bar{\mathbf{F}}(X_{n-k:n})}{\bar{\mathbf{F}}(U(n/k))}.$$

Using equations (4.8) and (4.9), we have

$$\mathbf{D}_n^* \approx \frac{\{1 - \exp(-\Lambda_n(X_{n-k:n}))\} - \{1 - \exp(-\Lambda(X_{n-k:n}))\}}{1 - \exp(-\Lambda(U(n/k)))}.$$

Since both $\bar{\mathbf{F}}_n(X_{n-k:n})$ and $\bar{\mathbf{F}}(X_{n-k:n})$ tend to zero in probability, then $\Lambda_n(X_{n-k:n})$ and $\Lambda(X_{n-k:n})$ go to zero in probability as well. Hence, by using the approximation $1 - \exp(-x) \sim x$, as $x \rightarrow 0$, we get

$$\mathbf{D}_n^* \approx \frac{\Lambda_n(X_{n-k:n}) - \Lambda(X_{n-k:n})}{\Lambda(U(n/k))} =: \mathbf{D}_n.$$

Now, we study the asymptotic behavior of \mathbf{D}_n . The numerator

$$\Lambda_n(X_{n-k:n}) - \Lambda(X_{n-k:n}) = - \int_{X_{n-k:n}}^{\infty} \frac{d\bar{\mathbf{F}}_n(z)}{C_n(z)} + \int_{X_{n-k:n}}^{\infty} \frac{d\bar{\mathbf{F}}(z)}{C(z)},$$

may be decomposed into $S_{n1} + S_{n2} + S_{n3}$, with

$$S_{n1} := - \int_{U(n/k)}^{\infty} \frac{d(\bar{\mathbf{F}}_n(z) - \bar{\mathbf{F}}(z))}{C(z)}, \quad S_{n2} := - \int_{X_{n-k:n}}^{\infty} \left\{ \frac{1}{C_n(z)} - \frac{1}{C(z)} \right\} d\bar{\mathbf{F}}_n(z),$$

and

$$S_{n3} := - \int_{X_{n-k:n}}^{U(n/k)} \frac{d(\bar{\mathbf{F}}_n(z) - \bar{\mathbf{F}}(z))}{C(z)}.$$

We will show that $\sqrt{k}S_{n1}/\Lambda(U(n/k))$ is an asymptotically centred Gaussian rv while both $\sqrt{k}S_{n2}/\Lambda(U(n/k))$ and $\sqrt{k}S_{n3}/\Lambda(U(n/k))$ tend to zero (in probability) as $n \rightarrow \infty$. An integration by parts yields that $S_{n1} = S_{n1}^{(1)} - S_{n1}^{(2)}$, where

$$S_{n1}^{(1)} := \frac{\bar{\mathbf{F}}_n(U(n/k)) - k/n}{C(U(n/k))}$$

and (with a change of variables)

$$S_{n1}^{(2)} := \int_1^\infty \frac{\overline{F}_n(zU(n/k)) - \overline{F}(zU(n/k))}{C^2(zU(n/k))} dC(zU(n/k)).$$

It is easy to verify that

$$\frac{\sqrt{k}S_{n1}^{(1)}}{\Lambda(U(n/k))} = \frac{k/n}{\Lambda(U(n/k))C(U(n/k))} \alpha_n(1),$$

where $\alpha_n(\cdot)$ is the uniform tail empirical process defined at the beginning of the proof of Theorem 2.1. From Lemma (6.2), we infer that

$$\sqrt{k}S_{n1}^{(1)}/\Lambda(U(n/k)) \approx \gamma\gamma_1^{-1}\alpha_n(1). \quad (5.20)$$

For the term $S_{n1}^{(2)}$, we have

$$\frac{\sqrt{k}S_{n1}^{(2)}}{\Lambda(U(n/k))} = \frac{\int_1^\infty \frac{C^2(U(n/k))}{C^2(zU(n/k))} \alpha_n\left(\frac{n}{k}\overline{F}(zU(n/k))\right) d\frac{C(zU(n/k))}{C(U(n/k))}}{(n/k)\Lambda(U(n/k))C(U(n/k))}.$$

From Lemma (6.1), we know that the function C is regularly varying at infinity with index $-1/\gamma_2$, then by using Potter's inequality, together with (6.26), we get

$$\frac{\sqrt{k}S_{n1}^{(2)}}{\Lambda(U(n/k))} \approx -(\gamma_1 + \gamma_2)^{-1} \int_1^\infty z^{1/\gamma_2-1} \alpha_n\left(\frac{n}{k}\overline{F}(zU(n/k))\right) dz,$$

which, by the change of variables $s = \frac{n}{k}\overline{F}(zU(n/k)) = \frac{n}{k}\overline{F}(zF^{\leftarrow}(1 - k/n))$, becomes

$$\frac{\sqrt{k}S_{n1}^{(2)}}{\Lambda(U(n/k))} \approx (\gamma_1 + \gamma_2)^{-1} \int_0^1 (\psi_n(s))^{1/\gamma_2-1} \alpha_n(s) d\psi_n(s),$$

where $\psi_n(s) := F^{\leftarrow}(1 - ks/n)/F^{\leftarrow}(1 - k/n)$. Making use, once again, of Potter's inequality of Lemma 6.3 to the quantile function $s \rightarrow F^{\leftarrow}(1 - s)$, yields

$$\frac{\sqrt{k}S_{n1}^{(2)}}{\Lambda(U(n/k))} \approx -\frac{\gamma_1\gamma_2}{(\gamma_1 + \gamma_2)^2} \int_0^1 s^{-\gamma/\gamma_2-1} \alpha_n(s) ds. \quad (5.21)$$

Subtracting (5.21) from (5.20) and using the weak approximation (5.13), we get

$$\frac{\sqrt{k}S_{n1}}{\Lambda(U(n/k))} \approx \frac{\gamma_1\gamma_2}{(\gamma_1 + \gamma_2)^2} \int_0^1 s^{-\gamma/\gamma_2-1} W_1(s) ds + \frac{\gamma}{\gamma_1} W_1(1) + o_p(1).$$

Note that the centred rv $\int_0^1 s^{-\gamma/\gamma_2-1} W_1(s) ds$ has a finite second moment (in fact it is equal to $2\gamma_2^2/((\gamma_2 - \gamma)(\gamma_2 - 2\gamma))$). As a result, the approximation above becomes

$$\frac{\sqrt{k}S_{n1}}{\Lambda(U(n/k))} = \frac{\gamma_1\gamma_2}{(\gamma_1 + \gamma_2)^2} \int_0^1 s^{-\gamma/\gamma_2-1} W_1(s) ds + \frac{\gamma}{\gamma_1} W_1(1) + o_p(1).$$

Now, we consider the second term S_{n2} . Since $\overline{F}_n(z) = 0$, for $z \geq X_{n:n}$, then

$$S_{n2} = \int_{X_{n-k:n}}^{X_{n:n}} \frac{C_n(z) - C(z)}{C_n(z)C(z)} d\overline{F}_n(z).$$

It follows that

$$|S_{n2}| \leq \theta_n \int_{X_{n-k:n}}^{\infty} \frac{|C_n(z) - C(z)|}{C^2(z)} dF_n(z),$$

where $\theta_n := \sup_{X_{1:n} \leq z \leq X_{n:n}} \{C(z)/C_n(z)\}$, which is stochastically bounded (see, e.g., [Stute and Wang, 2008](#)). We have $C = \overline{G} - \overline{F}$ and $C_n = \overline{G}_n - \overline{F}_n$, then $|S_{n2}| \leq \theta_n (T_{n1} + T_{n2})$, where

$$T_{n1} := \int_{X_{n-k:n}}^{\infty} \frac{|\overline{F}_n(z) - \overline{F}(z)|}{C^2(z)} dF_n(z) \quad \text{and} \quad T_{n2} := \int_{X_{n-k:n}}^{\infty} \frac{|\overline{G}_n(z) - \overline{G}(z)|}{C^2(z)} dF_n(z).$$

The set $\mathcal{A}_{n,\epsilon} := \{|X_{n-k:n}/U(n/k) - 1| > \epsilon\}$, $0 < \epsilon < 1$ is such that $\mathbf{P}(\mathcal{A}_{n,\epsilon}) \rightarrow 0$ as $n \rightarrow \infty$. For convenience, let $u_{n,\epsilon} := (1 - \epsilon)U(n/k)$ and

$$T_{n1}(\epsilon) := \int_{u_{n,\epsilon}}^{\infty} \frac{|\overline{F}_n(z) - \overline{F}(z)|}{C^2(z)} dF_n(z).$$

It is obvious that, for $\vartheta > 0$,

$$\mathbf{P}\left(\frac{\sqrt{k}T_{n1}}{\Lambda(U(n/k))} > \vartheta\right) \leq \mathbf{P}\left(\frac{\sqrt{k}T_{n1}(\epsilon)}{\Lambda(U(n/k))} > \vartheta\right) + \mathbf{P}(\mathcal{A}_n).$$

Then it remains to show that $\mathbf{P}\left(\frac{\sqrt{k}T_{n1}(\epsilon)}{\Lambda(U(n/k))} > \vartheta\right) \rightarrow 0$ as $n \rightarrow \infty$. To this end, let us write

$$\begin{aligned} \frac{\sqrt{k}T_{n1}(\epsilon)}{\Lambda(U(n/k))} &= \left\{ \frac{\overline{F}(U(n/k))}{C(U(n/k))} \right\} \left\{ \frac{k/n}{\Lambda(U(n/k))C(U(n/k))} \right\} \\ &\quad \times \left\{ \frac{C(U(n/k))}{C(u_{n,\epsilon})} \right\}^2 \int_1^{\infty} \frac{|\alpha_n(n\overline{F}(zu_{n,\epsilon})/k)|}{[C(zu_{n,\epsilon})/C(u_{n,\epsilon})]^2} d\frac{F_n(zu_{n,\epsilon})}{\overline{F}(U(n/k))} \end{aligned}$$

The regular variation property of C , that implies that $C(U(n/k))/C(u_{n,\epsilon}) \rightarrow (1 - \epsilon)^{1/\gamma_2}$, as $n \rightarrow \infty$, together with (5.13), (6.26) and Potter's inequality (see Lemma 6.3), give

$$\frac{\sqrt{k}T_{n1}(\epsilon)}{\Lambda(U(n/k))} = O_{\mathbf{P}}(1) \frac{\overline{F}(U(n/k))}{C(U(n/k))} \frac{\gamma}{\gamma_1} \int_1^{\infty} z^{2/\gamma_2} d\frac{F_n(zu_{n,\epsilon})}{\overline{F}(U(n/k))}.$$

The expectation of the integral in the previous equation equals

$$- \int_1^{\infty} z^{2/\gamma_2} d(\overline{F}(zu_{n,\epsilon})/\overline{F}(U(n/k))),$$

which, by routine manipulations and the fact that the parameters γ_1 and γ_2 are such that $\gamma_1 < \gamma_2$, converges to $-(1 - \epsilon)^{-1/\gamma} \gamma_2/(2\gamma - \gamma_2)$ as $n \rightarrow \infty$. On the other hand, we have $\overline{F}(U(n/k)) = k/n$ and $(k/n)/C(U(n/k)) \rightarrow 0$ as $n \rightarrow \infty$ (from Lemma 6.1). Therefore, $\sqrt{k}T_{n1}(\epsilon)/\Lambda(U(n/k)) \xrightarrow{\mathbf{P}} 0$ as $n \rightarrow \infty$ and so does $\sqrt{k}T_{n1}/\Lambda(U(n/k))$. Similar arguments lead to the same result for $\sqrt{k}T_{n2}/\Lambda(U(n/k))$,

therefore we omit details. Finally, we focus on the third term S_{n3} , for which an integration by parts yields

$$\begin{aligned} S_{n3} &= \int_{X_{n-k:n}}^{U(n/k)} \frac{\overline{F}_n(z) - \overline{F}(z)}{C^2(z)} dC(z) \\ &+ \frac{\overline{F}_n(X_{n-k:n}) - \overline{F}(X_{n-k:n})}{C(X_{n-k:n})} - \frac{\overline{F}_n(U(n/k)) - \overline{F}(U(n/k))}{C(U(n/k))}. \end{aligned}$$

Changing variables and using the process $\alpha_n(\cdot)$, we get

$$\begin{aligned} \frac{\sqrt{k}S_{n3}}{\Lambda(U(n/k))} &= \frac{k/n}{C(U(n/k))\Lambda(U(n/k))} \\ &\times \left\{ \int_{X_{n-k:n}/U(n/k)}^1 \frac{\alpha_n\left(\frac{n}{k}\overline{F}(zU(n/k))\right)}{[C(zU(n/k))/C(U(n/k))]^2} d\left(\frac{C(zU(n/k))}{C(U(n/k))}\right) \right. \\ &\quad \left. + \frac{C(U(n/k))}{C(X_{n-k:n})} \alpha_n\left(\frac{n}{k}\overline{F}(X_{n-k:n})\right) - \alpha_n(1) \right\}. \end{aligned}$$

For convenience, we set

$$\xi_n^+ := \max(X_{n-k:n}/U(n/k), 1) \text{ and } \xi_n^- := \min(X_{n-k:n}/U(n/k), 1).$$

By using routine manipulations, including Potter's inequality (see Lemma 6.3) and the fact that $\sup_{0 < t < 1} \alpha_n(t)$ is stochastically bounded, we show that

$$\frac{\sqrt{k}S_{n3}}{\Lambda(U(n/k))} \approx \frac{\gamma}{\gamma_1} \left\{ O_{\mathbf{P}}(1) (\xi_n^+ - \xi_n^-) + \frac{C(U(n/k))}{C(X_{n-k:n})} \alpha_n\left(\frac{n}{k}\overline{F}(X_{n-k:n})\right) - \alpha_n(1) \right\}.$$

Since $\xi_n^+ - \xi_n^- = |1 - X_{n-k:n}/U(n/k)|$ and $X_{n-k:n}/U(n/k) \xrightarrow{\mathbf{P}} 1$, then $\xi_n^+ - \xi_n^- \xrightarrow{\mathbf{P}} 0$.

Now, it is clear that

$$\begin{aligned} \frac{\sqrt{k}S_{n3}}{\Lambda(U(n/k))} &= \frac{\gamma}{\gamma_1} \left\{ \frac{C(U(n/k))}{C(X_{n-k:n})} \left(\alpha_n\left(\frac{n}{k}\overline{F}(X_{n-k:n})\right) - \alpha_n(1) \right) \right. \\ &\quad \left. + \left(\frac{C(U(n/k))}{C(X_{n-k:n})} - 1 \right) \alpha_n(1) \right\} + o_{\mathbf{P}}(1). \end{aligned}$$

We have $C(U(n/k))/C(X_{n-k:n}) \xrightarrow{\mathbf{P}} 1$ and $\alpha_n(1) = O_{\mathbf{P}}(1)$, then it suffices to show that $\alpha_n\left(\frac{n}{k}\overline{F}(X_{n-k:n})\right) - \alpha_n(1) \xrightarrow{\mathbf{P}} 0$. Indeed, making use of the approximation (5.13), we get

$$\alpha_n\left(\frac{n}{k}\overline{F}(X_{n-k:n})\right) - \alpha_n(1) = W_1\left(\frac{n}{k}\overline{F}(X_{n-k:n})\right) - W_1(1) + o_{\mathbf{P}}(1).$$

Since $\{W_1(t), 0 \leq t \leq 1\}$ is a Wiener process, then it is easy to verify that

$$\left| W_1\left(\frac{n}{k}\overline{F}(X_{n-k:n})\right) - W_1(1) \right| \stackrel{d}{=} \left| W_1\left(\left|\frac{n}{k}\overline{F}(X_{n-k:n}) - 1\right|\right) \right|.$$

Recall that $\frac{n}{k}\overline{F}(X_{n-k:n}) \xrightarrow{P} 1$, then by using similar arguments as those used in the proof of Lemma 5.2 (i) in [Brahim et al. \(2014\)](#), we show that $W_1 \left(\left| \frac{n}{k}\overline{F}(X_{n-k:n}) - 1 \right| \right)$ tends to zero in probability, which implies that $\sqrt{k}S_{n3}/\Lambda(U(n/k)) \xrightarrow{P} 0$ as well. In summary, we showed that

$$\sqrt{k}\mathbf{D}_n^* = \frac{\gamma_1\gamma_2}{(\gamma_1 + \gamma_2)^2} \int_0^1 s^{-\gamma/\gamma_2-1} W_1(s) ds + \frac{\gamma}{\gamma_1} W_1(1) + o_p(1),$$

which leads to the wanted result. Finally, with some elementary calculations, we get the variance of the Gaussian variable $\sqrt{k}(\overline{\mathbf{F}}_n(X_{n-k:n})/\overline{\mathbf{F}}(X_{n-k:n}) - 1)$ and conclude the proof. \square

5.5. Proof of Theorem 4.2. For the sake of notational simplicity, we set $\ell = \ell_n := U(n/k)$. Let us rewrite Π into

$$\Pi = u\overline{\mathbf{F}}(u) \int_1^\infty \frac{\overline{\mathbf{F}}(ux)}{\overline{\mathbf{F}}(u)} dx,$$

and consider the decomposition

$$\frac{\widehat{\Pi}_n - \Pi}{(u/\ell)^{1-1/\gamma_1} \ell \overline{\mathbf{F}}(\ell)} = \sum_{i=1}^7 S_{ni},$$

where

$$\begin{aligned} S_{n1} &:= \left\{ \frac{(u/X_{n-k:n})^{1-1/\widehat{\gamma}_1}}{(u/\ell)^{1-1/\gamma_1}} - 1 \right\} \frac{\widehat{\gamma}_1}{1 - \widehat{\gamma}_1} \frac{X_{n-k:n}}{\ell} \frac{\overline{\mathbf{F}}_n(X_{n-k:n})}{\overline{\mathbf{F}}(\ell)}, \\ S_{n2} &:= \frac{X_{n-k:n}}{\ell} \frac{\overline{\mathbf{F}}_n(X_{n-k:n})}{\overline{\mathbf{F}}(\ell)} \left\{ \frac{\widehat{\gamma}_1}{1 - \widehat{\gamma}_1} - \frac{\gamma_1}{1 - \gamma_1} \right\}, \\ S_{n3} &:= \frac{\gamma_1}{1 - \gamma_1} \frac{\overline{\mathbf{F}}(X_{n-k:n})}{\overline{\mathbf{F}}(\ell)} \frac{\overline{\mathbf{F}}_n(X_{n-k:n})}{\overline{\mathbf{F}}(X_{n-k:n})} \left\{ \frac{X_{n-k:n}}{\ell} - 1 \right\}, \\ S_{n4} &:= \frac{\gamma_1}{1 - \gamma_1} \frac{\overline{\mathbf{F}}_n(X_{n-k:n})}{\overline{\mathbf{F}}(X_{n-k:n})} \left\{ \frac{\overline{\mathbf{F}}(X_{n-k:n})}{\overline{\mathbf{F}}(\ell)} - \left(\frac{X_{n-k:n}}{\ell} \right)^{-1/\gamma_1} \right\} \\ S_{n5} &:= \frac{\gamma_1}{1 - \gamma_1} \frac{\overline{\mathbf{F}}_n(X_{n-k:n})}{\overline{\mathbf{F}}(X_{n-k:n})} \left\{ \left(\frac{X_{n-k:n}}{\ell} \right)^{-1/\gamma_1} - 1 \right\}, \\ S_{n6} &:= \frac{\gamma_1}{1 - \gamma_1} \left\{ \frac{\overline{\mathbf{F}}_n(X_{n-k:n})}{\overline{\mathbf{F}}(X_{n-k:n})} - 1 \right\}, \\ S_{n7} &:= \frac{\gamma_1}{1 - \gamma_1} - \left(\frac{u}{\ell} \right)^{1/\gamma_1} \frac{\overline{\mathbf{F}}(u)}{\overline{\mathbf{F}}(\ell)} \int_1^\infty \frac{\overline{\mathbf{F}}(ux)}{\overline{\mathbf{F}}(u)} dx. \end{aligned}$$

We start by representing the five quantities $\sqrt{k}S_{ni}$, $i = 1, 2, 3, 5, 6$ in terms of the Gaussian processes W_1 and W_2 , given in Theorem 2.1, then we show that $\sqrt{k}S_{n4}$ and $\sqrt{k}S_{n7}$ converge to deterministic limits. For the first term S_{n1} , recall that

$X_{n-k:n} \approx \ell$, which implies by the regular variation of $\bar{\mathbf{F}}$ that $\bar{\mathbf{F}}(X_{n-k:n}) \approx \bar{\mathbf{F}}(\ell)$. On the other hand, we have $\hat{\gamma}_1 \xrightarrow{\mathbf{P}} \gamma_1$ and, from Remark 4.1, $\bar{\mathbf{F}}_n(X_{n-k:n}) \approx \bar{\mathbf{F}}(X_{n-k:n})$. It follows that

$$S_{n1} \approx \frac{\gamma_1}{1 - \gamma_1} \left\{ \frac{(u/X_{n-k:n})^{1-1/\hat{\gamma}_1}}{(u/\ell)^{1-1/\gamma_1}} - 1 \right\} = \frac{\gamma_1}{1 - \gamma_1} \left\{ S_{n1}^{(1)} + S_{n1}^{(2)} \right\},$$

where $S_{n1}^{(1)} := (u/\ell)^{1/\gamma_1 - 1/\hat{\gamma}_1} - 1$ and $S_{n1}^{(2)} := (u/\ell)^{1/\gamma_1 - 1/\hat{\gamma}_1} \left((\ell/X_{n-k:n})^{1-1/\hat{\gamma}_1} - 1 \right)$.

By using the mean value theorem in $S_{n1}^{(1)}$, we have

$$S_{n1}^{(1)} = (1/\gamma_1 - 1/\hat{\gamma}_1) (u/\ell)^{\epsilon_n} \log(u/\ell),$$

with ϵ_n being between $1/\gamma_1 - 1/\hat{\gamma}_1$ and 0. The consistency of $\hat{\gamma}_1$ implies that $\epsilon_n \xrightarrow{\mathbf{P}} 0$, and therefore $S_{n1}^{(1)} \approx \gamma_1^{-2} (\hat{\gamma}_1 - \gamma_1) \log(u/\ell)$. Likewise, we may readily show that

$$S_{n1}^{(2)} \approx \frac{1 - \gamma_1}{\gamma_1} \left(\frac{X_{n-k:n}}{\ell} - 1 \right).$$

Consequently,

$$S_{n1} \approx \frac{\log(u/\ell)}{\gamma_1(1 - \gamma_1)} (\hat{\gamma}_1 - \gamma_1) + \left(\frac{X_{n-k:n}}{\ell} - 1 \right).$$

By using similar arguments we also show that

$$S_{n2} \approx \frac{\hat{\gamma}_1 - \gamma_1}{(1 - \gamma_1)^2}, \quad S_{n3} \approx \frac{\gamma_1}{1 - \gamma_1} \left(\frac{X_{n-k:n}}{\ell} - 1 \right) \quad \text{and} \quad S_{n5} \approx -\frac{1}{1 - \gamma_1} \left(\frac{X_{n-k:n}}{\ell} - 1 \right).$$

Summing these four terms, we obtain

$$S_{n1} + S_{n2} + S_{n3} + S_{n5} \approx \frac{(1 - \gamma_1) \log(u/\ell) + \gamma_1}{\gamma_1(1 - \gamma_1)^2} (\hat{\gamma}_1 - \gamma_1).$$

Now, we use the second approximation in Theorem 2.1 to have

$$\begin{aligned} \sqrt{k} (S_{n1} + S_{n2} + S_{n3} + S_{n5}) &\approx \frac{(1 - \gamma_1) \log a + \gamma_1}{\gamma_1(1 - \gamma_1)^2} \\ &\times \left\{ \int_0^1 t^{-1} (cW_1(t) - c_2W_2(t)) dt - cW_1(1) + c_2W_2(1) + \mu(k) + o_{\mathbf{P}}(1) \right\}. \end{aligned} \quad (5.22)$$

The asymptotic representation of Theorem 4.1 yields

$$\sqrt{k} S_{n6} \approx \frac{\gamma_1}{1 - \gamma_1} \left\{ \frac{\gamma_1 \gamma_2}{(\gamma_1 + \gamma_2)^2} \int_0^1 s^{-\gamma/\gamma_2 - 1} W_1(s) ds + \frac{\gamma}{\gamma_1} W_1(1) \right\} + o_{\mathbf{P}}(1). \quad (5.23)$$

For the fourth term S_{n4} , it suffices to use the second-order condition of regular variation (4.11) and the fact that $X_{n-k:n} \approx \ell$, to get

$$\sqrt{k} S_{n4} = o_{\mathbf{P}} \left(\sqrt{k} \mathbf{A}(\ell) \right) = o_{\mathbb{P}}(1), \quad \text{as } n \rightarrow \infty. \quad (5.24)$$

For the last term S_{n7} , we first note that

$$S_{n7} = \int_1^\infty x^{-1/\gamma_1} dx - \frac{u \bar{\mathbf{F}}(u)}{(u/\ell)^{1-1/\gamma_1} \ell \bar{\mathbf{F}}(\ell)} \int_1^\infty \frac{\bar{\mathbf{F}}(ux)}{\bar{\mathbf{F}}(u)} dx,$$

In addition to the regular variation of $|\mathbf{A}|$, we apply the uniform inequality of regularly varying functions (see, e.g., Theorem 2.3.9 in [de Haan and Ferreira, 2006](#), page 48) to show that

$$\sqrt{k}S_{n7} \sim \frac{\sqrt{k}\mathbf{A}(\ell)}{(\gamma_1 - 1 - \tau_1)(\gamma_1 - 1)}. \quad (5.25)$$

Finally, gathering results (5.22), (5.23), (5.24) and (5.25) yields a Gaussian approximation from which we derive the normal limiting distribution of the premium estimator $\hat{\Pi}_n$. Tedious computations for the asymptotic variance complete the proof of the theorem. \square

Concluding notes

We proposed an estimator of the tail index for randomly truncated heavy-tailed data based on the same number of extreme observations from both truncated and truncation variables. Thus, the determination of the optimal sample fraction becomes standard, in the sense of applying any convenient algorithm available in the literature. The asymptotic normality of the estimator is established by taking into account the dependence structure of the observations and a practical way to construct confidence bounds for the extreme value index is given. The obtained Gaussian approximations are of great usefulness as they allow to determine the limiting distributions of several statistics related to the extreme value index such that high quantiles and risk measures estimators (see, for instance, [Necir and Meraghni, 2009](#)). As an application, we provided an estimator for the excess-of-loss reinsurance premium in the case of large randomly truncated claims.

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6. APPENDIX

Lemma 6.1. *Assume that the second-order conditions (2.7) hold with $\gamma_1 < \gamma_2$. Then the function C is regularly varying at infinity with index $-1/\gamma_2$ and $t^{-1}C(U(t)) \rightarrow 0$ as $t \rightarrow \infty$.*

Proof. We have $C = \overline{G} - \overline{F}$ with $\gamma_1 < \gamma_2$, hence $C(x) \sim \overline{G}(x)$ as $x \rightarrow \infty$. Since both \overline{F} and \overline{G} satisfy the second-order conditions (2.7), then in view of Lemma 3 in Hua and Joe (2011), there exist two constants $\delta, \delta_2 > 0$, such that $\overline{F}(x) \sim \delta x^{-1/\gamma}$ and $\overline{G}(x) \sim \delta_2 x^{-1/\gamma_2}$, as $x \rightarrow \infty$. The first equivalence implies that $U(t) \sim \delta^\gamma t^\gamma$, as $t \rightarrow \infty$, therefore $C(U(t)) \sim \delta_2 \delta^{\gamma/\gamma_2} t^{\gamma/\gamma_2}$, it follows that

$$t^{-1}C(U(t)) \sim \delta_2 \delta^{\gamma/\gamma_2} t^{\gamma/\gamma_2 - 1}.$$

By assumption, we have $\gamma_1 < \gamma_2$, it follows that $\gamma/\gamma_2 = \gamma_1/(\gamma_1 + \gamma_2)$ is less to $1/2$, thus $t^{-1}C(U(t)) \rightarrow 0$ as $t \rightarrow \infty$, which achieves the proof of the lemma. \square

Lemma 6.2. *Under the assumptions of Lemma 6.1, we have*

$$t\Lambda(U(t))C(U(t)) \rightarrow \gamma_1/\gamma \text{ as } t \rightarrow \infty. \quad (6.26)$$

Proof. Recalling that $\Lambda(x) = \int_x^\infty dF(z)/C(z)$ and $\overline{F}(U(t)) = t^{-1}$, we write

$$t\Lambda(U(t))C(U(t)) = - \int_1^\infty \frac{C(U(t))}{C(zU(t))} \frac{d\overline{F}(zU(t))}{\overline{F}(U(t))}.$$

Making use of Potter's inequality for both C and \overline{F} , we infer that,

$$t\Lambda(U(t))C(U(t)) \sim - \int_1^\infty z^{1/\gamma_2} dz^{-1/\gamma} = \gamma_1/\gamma, \text{ as } t \rightarrow \infty,$$

as sought. \square

Lemma 6.3. *Suppose that φ is a regularly varying function (at infinity) with index $\rho \in \mathbb{R}$, i.e. $\varphi(tx)/\varphi(t) \rightarrow x^\rho$, as $t \rightarrow \infty$, for all $x > 0$. Then for any $0 < \epsilon < 1$, there exists $t_0 = t_0(\epsilon)$ such that for $t \geq t_0$, $tx \geq t_0$,*

$$(1 - \epsilon) x^\rho \min(x^\epsilon, x^{-\epsilon}) < \frac{\varphi(tx)}{\varphi(t)} < (1 + \epsilon) x^\rho \max(x^\epsilon, x^{-\epsilon}).$$

In other words, we have, for every $x_0 > 0$,

$$\lim_{t \rightarrow \infty} \sup_{x \geq x_0} \left| \frac{\varphi(tx)}{\varphi(t)} - x^\rho \right| = 0.$$

Proof. This result, known as Potter's bound inequalities, is stated in, for instance, [de Haan and Ferreira, 2006](#), Proposition B.1.9, Assertion 5, page 367. \square